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ABSTRACT

This paper considers how the algebra curriculum in secondary mathematics might be reformulated by rethinking both the content of algebra and the approaches to teaching and learning algebraic concepts. It considers how the concept of function can be made a central theme of the algebra curriculum and further suggests that computer software, when designed properly and used with appropriate materials, can provide an opportunity for engaging students in significant mathematical inquiry. After a discussion of the multiple representations of the concept of function, several facets of curriculum innovation involving computers are presented. Some lessons learned from cognitive research involving students' interactions with a variety of graphically based software environments are given. Three examples of software tools growing out of research findings involving visual representations of functions are presented; they are: "The Function Supposer"; "The Function Comparator"; and "The Function Analyzer." Examples of classroom materials developed to engage students in problem-solving and problem-finding activities and to take advantage of new opportunities for learning and teaching made available by the new computer software are presented. Finally, new directions in software environments beyond visualization are explored. One of these, a tool called "controlled dynamic phenomena" is described as an approach that will be a catalyst for new intuitions and understandings of functions. (32 references) (MDH)

REPORTS AND PAPERS IN PROGRESS

Improving the Teaching and Learning of Algebra Using a Visual Approach

Rethinking Algebra in Secondary Mathematics Education

Wayne Harvey

Report No. 91-2

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Improving the Teaching and Learning of Algebra Using a Visual Approach

Rethinking Algebra in Secondary Mathematics Education

Wayne Harvey

This paper considers how the algebra curriculum in secondary mathematics might be reformulated by rethinking both the content of algebra and the approaches to teaching and learning algebraic concepts. The author considers how the concept of function can be made a central theme of the algebra curriculum and further suggests that computer software, when designed properly and used with appropriate materials can provide an opportunity for engaging students in significant mathematical inquiry.

Algebra is a central focus in the secondary mathematics curriculum as it is currently structured. For many students, their first year of algebra is also their last year of mathematics. For those students who do continue, geometry often is cast as a kind of interlude between algebra 1 and algebra 2—a break from the routine, and one that is not destined to play much of a role in the courses that follow. The second year of algebra is partly a refurbishing of algebra 1 skills that have remained unused for a year, supplemented with greater emphasis on graphs of higher-order polynomials and transformations on these graphs (translations and dilations). Other topics, which vary from text to text, often include preparations for a second preparation (precalculus) before calculus.

Despite its position as the central thread in the mathematics curriculum, little in the way of an organizing idea *behind* the algebra curriculum is apparent to students or teachers. The lack of an organizing idea leads many to talk about the algebra curriculum as a list of skills to be learned, almost as if they were reciting the chapters in a typical algebra textbook. To students, and often enough to teachers as well, the algebra curriculum has no internal coherence—it appears as a collection of techniques for manipulating expressions.

The NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) present us with the beginnings of an appropriate resolution to this incoherence. One section in the *Standards* is devoted to a central theme in secondary mathematics: Functions (Standard 6 in "Curriculum Standards for Grades 9-12").

> The concept of function is an important unifying idea in mathematics. Functions, which are special correspondences between the elements of two sets, are common throughout the curriculum. In arithmetic, functions appear as the usual operations on numbers . . . ; in algebra, functions are relationships between variables that represent numbers; in geometry, functions relate sets of points to their images under motions such as flips, slides, and turns; and in probability, they relate events to their likelihoods. (p. 154)

There is lively interest now in how students come to understand functions, how well they translate among symbolic, graphical, tabular, and other representations of these functions; and what role functions play in the overall picture of a secondary mathematics education.

One underlying assumption of this paper, then, is that the concept of function can and should be a central theme of the algebra curriculum. It is clear that the notion of function is at the foundation of elementary calculus, and a facile understanding of and ability to use both graphical and symbolic representations of functions are critical for work in almost all scientific disciplines. Unfortunately, few high school students (or even college students completing calculus) are able to identify equivalence between algebraic and graphical representations of functions, interpret graphs accurately, or develop an intuitive understanding of functions and their representation as graphs (Fey 1984; Clement 1985; Goldenberg 1987; Goldenberg & Kliman 1990; Eisenberg & Dreyfus 1991).

The NCTM *Standards* (1989) also emphasize that—in the shift towards developing students' understanding of functions and their various representations—appropriate uses of computers and graphing calculators should be explored.

> Computing technology provides tools, especially spreadsheets and graphing utilities, that make the study of function concepts and their applications accessible to all students in grades 9-12.
>
> The integration of ideas from algebra and geometry is particularly strong, with graphical representation playing an important connecting role. Thus, frequent reference to graphing utilities will be found throughout these standards; by this we mean a computer with appropriate graphing software. . . . (p. 125)

But the *Standards* do not give us insight into what "appropriate" graphing software might be. Although there is widespread agreement that computers can and should play a significant role in a new algebra curriculum, there is not similar agreement about what constitutes *effective* uses of computers in the teaching and learning of algebra. This is an important area for future research, and this paper describes some preliminary attempts to build and explore software environments for studying algebraic concepts which may inform future work. The focus of this work has been on *function* and *representation*.

"Function" and "Representation"

Functions may be described by a string of symbols such as $f(x) = x^2 - 3x + 6$, represented by a graph or table, explained in words, or implemented as a computer procedure. Only such manifestations are available for us to manipulate and study. Functions, themselves, are an abstraction—at the heart of the matter, but invisible. It is likely that viewing functions from more than one of these perspectives will ultimately build a more robust concept than can readily be built from one perspective alone; that assumption seems to be the basis of the NCTM's position as stated above. Since it is too time consuming or too limiting to create and work with most representations by hand, computers are necessary tools for fully interacting with and manipulating multiple representations.

To make sense out of multiple representations of a single underlying object—the function under consideration—one must be able to translate readily among the representations and, more importantly, reconcile the different information provided by the different representations so as to understand the common abstraction underlying all of them. Students' difficulty in doing this is well attested (Clement 1982; Kaput & Sims-Knight 1983, Eisenberg & Dreyfus 1991; Goldenberg & Kliman 1990).

Software tools that use *multiple linked representations* hold great potential for helping students understand the concept of function. The use of *linked* representations—where changes in one representation are automatically and immediately reflected in alternative representations—can contribute significantly to the development of a flexible understanding of the notion of function. This method can help students learn to see these representations as alternative views of the same underlying object—the function.

Of course, software alone will never bring about major improvements in the teaching and learning of algebra. No single ingredient, acting alone, will suffice. However, software can contribute to and support changes in teaching and learning algebra. To do so, software must be developed in the context of ongoing cognitive research, must address teacher support needs, and must be accompanied by materials and activities that effectively draw upon the pedagogical principles of the tool.

The Many Facets of Curriculum Innovation Involving Computers

At Education Development Center (EDC), an early attempt at achieving this kind of integration was carried out in geometry, with *The Geometric Supposer* software (Schwartz &Yerushalmy 1985-1991) and curriculum series (Chazan & Houde 1989; Yerushalmy & Chazan in press).[1] The software has been accompanied by project activities and teacher support materials, including video. All are designed to be part of a comprehensive approach to meeting teacher and student needs. This approach has been advocated by a variety of researchers (for example, Dugdale 1981; Kelman et al. 1983; Fey 1984; Kaput 1986) and is also embodied in the *Visualizing Algebra* software environments developed by me and my colleagues at EDC and published by Sunburst Communications (Harvey et al. 1989; Mark & Harvey 1990; Harvey et al. 1990).

These initial efforts have suggested the effectiveness of integrating the following strands of work as we rethink the way algebra is taught and learned in secondary school classrooms:

- conducting research on students' interpretation of, and interactions with, visual environments for exploring functions (Goldenberg & Harvey 1989-1991; Goldenberg & Kliman 1990; Goldenberg 1988; Goldenberg et al. 1987)
- developing multiple representation software allowing students to manipulate functions visually (Harvey et al. 1990; Harvey et al. 1989; Goldenberg & Harvey 1989-1991)
- developing project-based curriculum materials approaching algebra learning visually (Harvey et al. 1989; Goldenberg & Harvey 1989-1991)
- working with teachers in high school classrooms to integrate these new approaches to algebra teaching and learning into the existing curriculum (Ruopp 1990-1992)

This paper describes some of our research efforts and findings, discusses some of the software environments that grew out of this research, and, finally, presents some examples of the curriculum materials that are so critical for successfully integrating software into the classroom. Our efforts to work with teachers to assist them in adapting to such new approaches are discussed elsewhere (Chazan & Houde 1989; Yerushalmy et al. 1990; Yerushalmy & Houde 1985).

Some Lessons from the Research

While developing, testing, and refining our algebra software, and in preparation for developing appropriate curricular materials for use with such software, we performed some basic cognitive research involving students' interactions with a variety of graphically based software environments.[2] Teachers, mathematicians, and mathematics education specialists helped analyze transcripts which recorded students' work with software prototypes and algebra activities designed specifically to elucidate some of our conjectures about students' understandings and misunderstandings. While much of this work is described in detail elsewhere (Goldenberg & Kliman 1990; Goldenberg 1990; Goldenberg 1988; Goldenberg 1987; Goldenberg et al. 1987), included below are a few examples of how the research informed the development and revision of the software and led to new insights into the design of curriculum and teaching methods.

Our work has shown students to have some very weak notions—sometimes very incorrect notions—of the function concept. These previous conceptions often determine to a great extent what students will see, or not see, when interacting with visually based software.

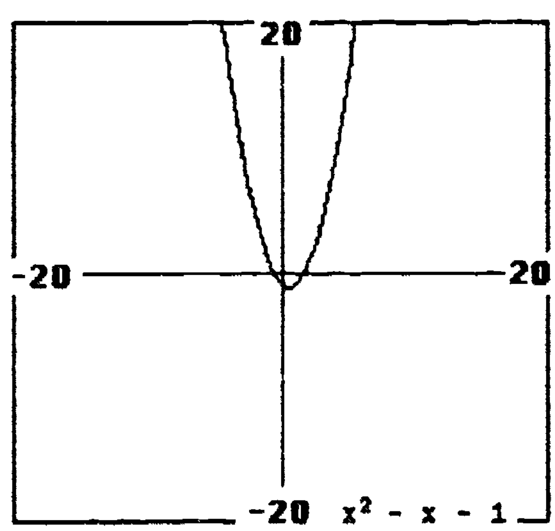
For example, we have found that students' perceptions when they view graphs may differ from those of their teachers, and the strategies with which students act on their perceptions are often limited or confused (Goldenberg & Kliman 1990; Goldenberg 1990; Goldenberg 1988; Goldenberg et al. 1987). One example of this perceptual confusion is described below.

> When students look only at the symbolic representation of a function such as a second-order polynomial, they often easily recognize that the domain is unlimited: any value may be plugged in for x and a value for y may be computed. On the other hand, their visual impression of the graph of the same function often takes precedence over symbolic analysis and leads students to reason . . . as if the domain is bounded somewhere within the extremes of the domain depicted in the graph. In a graph such as the one in [the figure opposite], x "looks like" it will never grow beyond roughly ± 10. (Goldenberg 1988, p. 162)

This is the kind of research finding that has important implications for the design of educational graphing software and related curriculum materials. In this case, it demonstrated the need to place greater emphasis on scaling issues. We took this into account as we designed *The Function Analyzer* software, described in the next section.

Perhaps the most important conclusion of our research is that *educational* uses of graphing tools place very different demands on the software than scientific or engineering uses do.[3] This is to be expected. When engineers and scientists use graphers, they are often interested primarily in the behavior of a *particular* function. Students, too, must deal with particular functions, but most of the educational value is in the generalizations students abstract from the particulars. The shape of $-2x^2 + 30x - 108$ is of no special educational consequence, but it may serve as a data point about any of several broader classes: a particular family of quadratics (for example, ones that differ only in the constant term); more generally all quadratics; still more generally all polynomials; or even all functions.

This difference in purpose has particular implications for the user interface. If software is intended for student use, it must be easy to modify functions—not just through a general editor of algebraic expressions, but through a "smart" editor that understands the syntax of the expression and allows a student easily to find and increment or decrement some parameter in the expression. For example, when a student is interested in exploring the role of the



Illusion of Constrained Domain

linear coefficient in a function defined as $f(x) = x^2 - 6x + 3$ — a manipulation which, conceptually, involves merely stepping up from -6 in increments—a smart editor saves having to attend to delete keys, spaces, signs, and potential typographical errors. All of our algebra software tools allow for this kind of direct manipulation of coefficients in expressions, thereby permitting students to devote more attention to the nature of the experiment and its outcome.

Our research also showed, however, that experiments of this kind can lead to unexpected confusions in a student's understanding of function and, more specifically, the student's emerging concept of variable (Goldenberg 1988). For students to interpret graphical representations of algebraic expressions correctly, they must understand the meaning of the variable and of the parameters of the function that surround it (for example, coefficients in a polynomial). To the naive student, there is little difference between the a, b, c, and x in the definition $f(x) = ax^2 + bx + c$. All four letters' values seem equally "variable." Yet, as the function is defined, only x is *its* variable. Further, when a student studies quadratics, this form is a stand-in for an entire *class* of functions. As students explore the effects of varying the values of a, b, and c on the graph of the function $f(x)=ax^2+bx+c$, they are really studying not $f(x)$, whose variable, x, is numeric and whose output is numeric, but some different kind of function, $F(a,b,c)$, whose three variables a. b, and c are numeric and whose output (a *particular* quadratic function) is visualized by its graph. The concept of variable is already difficult for students to learn; yet, here we see variables and constants switching roles as students experiment with graphing. Our latest software experiments attempt to address these concerns by including some new ways for students to interact with graphical representations. This work is described at the end of this paper in the section "New Directions."

As noted earlier, any particular function is not what algebra is about. Rather, it is only as students abstract important features of whole *families* of functions that they develop the kind of knowledge and intuition on which they can build. Helping students make such abstractions is the role of the teacher, assisted by appropriate curriculum materials. Consider the task of helping students to understand the essential notion of variable when studying functions like $F(b) = x^2 + bx + 2$ whose domain variable is a real number but whose range elements are themselves functions with both domain and range in the real numbers. In the context of the conventional curriculum, these new objects—function-valued functions—may seem terribly abstruse, but the ideas are apparently quite natural and are even reflected in much of our language. When we make statements like "$ax^2 + 5$ generates an upward-opening parabola with its vertex above the origin when a is positive," we are, in effect, describing a non-numeric "value" (the shape of the picture) of a function of a. Curriculum and teaching approaches can capitalize on such intuitive understandings.

In general, where the research we conducted showed that capabilities provided by the computer can lead to unexpected pitfalls in students' understanding, we have tried both to take these pitfalls into account in our software design and to develop curriculum materials and identify teaching strategies that address the pitfalls.

Software Environments for Studying the Concept of Function

In collaboration with algebra teachers, mathematicians, cognitive researchers, and software developers, we have designed several software tools that use multiple linked representations. In testing our software with students, we have found reason to be optimistic that students can acquire significantly deeper understandings of the function concept by working more directly with carefully designed visual representations of functions. This section describes three such software tools and provides examples of their use.

The Function Analyzer. *The Function Analyzer* provides tools for the exploration and manipulation of functions as expressions or graphs. This software allows one to manipulate the function expression or the function graph, examine the values of the function, plot points on the coordinate plane, or change the scale of the coordinate plane. Functions are plotted on three related grids (figure 1, page 6): (1) the large view on the left side of the screen, (2) the "zoomed out" view on the upper right portion

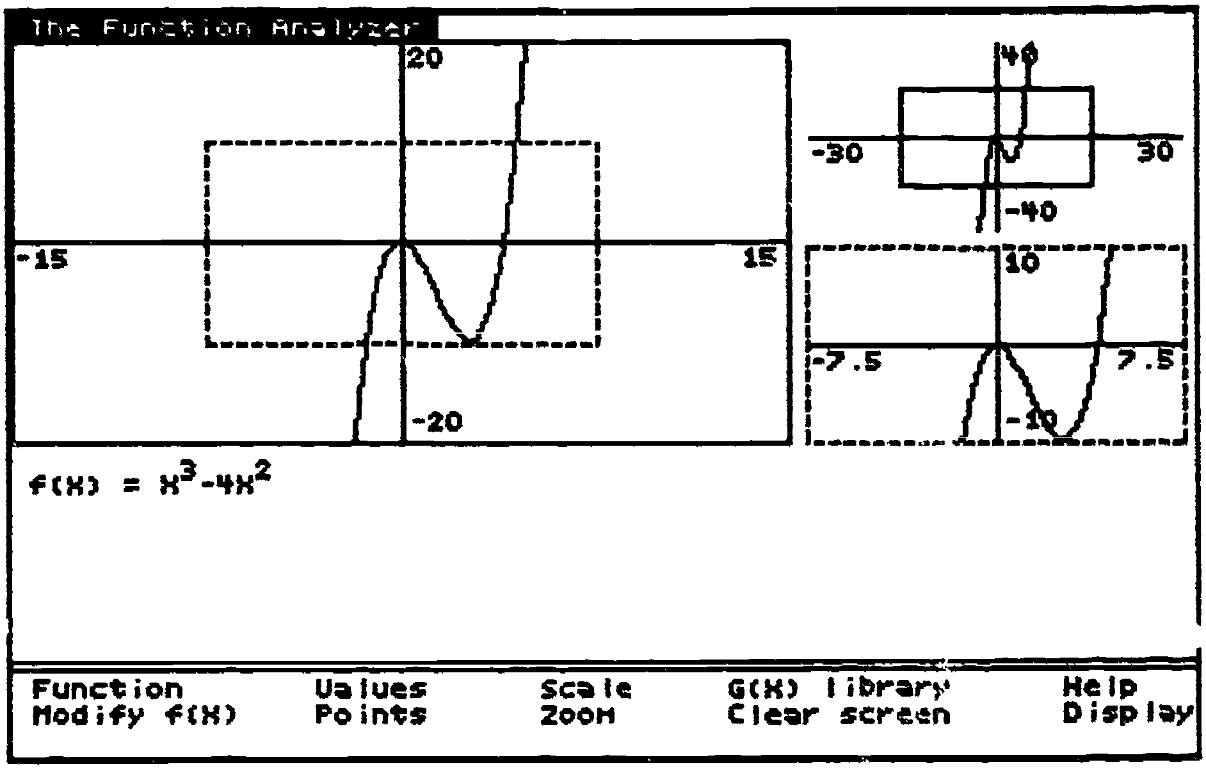


Figure 1. *The Function Analyzer.*

of the screen, and (3) the "zoomed in" view on the lower right portion of the screen. By presenting each function at three different scales, the software focuses attention on the importance of scale in interpreting the graphical representation.

One can use the software to plot a variety of functions and then change them in a number of ways. One can vary the symbolic parameters of a function, for example, changing the -6 in the function $x^2 - 6x + 3$ to the values -6, -4, -2, 0, 2, 4, and 6. This generates a family of curves that provides some insight into the role of the linear term in the quadratic function. As the coefficient of the linear term is varied, the parabola moves up one side of the screen and then down the other side, without changing shape. Figure 2 shows the results of this sequence of graphs.

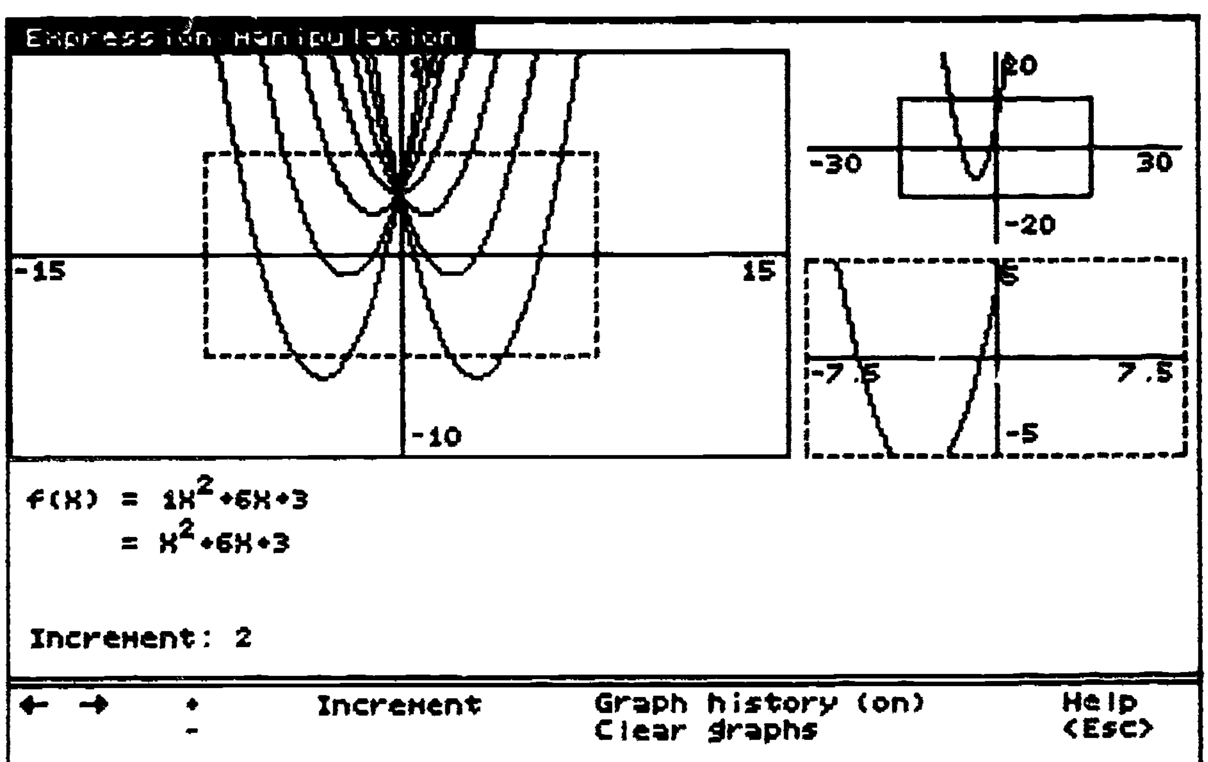


Figure 2. The family of functions, $f_b(x) = x^2 + bx + 3$.

We have been especially interested in allowing students to manipulate representations of functions other than symbolic ones. To this end, we have designed the software to allow students to manipulate function graphs directly by translating, stretching, and reflecting them. Translating a graph (figure 3, opposite) corresponds in the symbolic world either to substituting x + *constant* for x or to adding a constant to $f(x)$. Stretching a graph corresponds either to substituting ax for x or to multiplying $f(x)$ by a constant. Reflecting a graph corresponds either to substituting $-x$ for x or to multiplying $f(x)$ by -1. These features allow students to explore functions as *mathematical objects* that can be manipulated and acted upon rather than strictly as processes that serve to generate values.

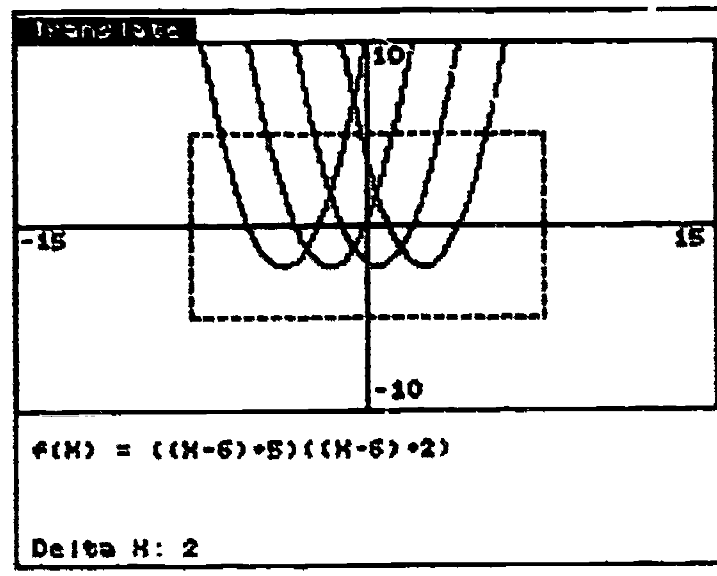


Figure 3. Translating the graph of $f(x) = (x+5)(x+2)$.

Of course it is important for students also to recognize function expressions as recipes for generating values. In an effort to more clearly relate the values of x to the values of $f(x)$, and the pair of values $(x, f(x))$ to a point on the graph, the software provides a table of values as a third representation of function that is linked to the graphical representation by both color coding and a special bar on the graph (figure 4, below). The values table displays the values for x, $f(x)$, and $g(x)$. A vertical bar connecting the current x-value with its corresponding function values is displayed in the graph window.

The Function Supposer. Whereas *The Function Analyzer* is an environment in which students can explore individual functions, their various representations, and function families, a second software environment — *The Function Supposer* — allows students to compose new functions by adding, subtracting, multiplying, and dividing functions, both symbolically and graphically (Harvey et al. 1990). For example, figure 5 (page 8) shows two functions being combined to form a third function. The operation, in which $h(x) = f(x) \cdot g(x)$, is displayed both symbolically and graphically.

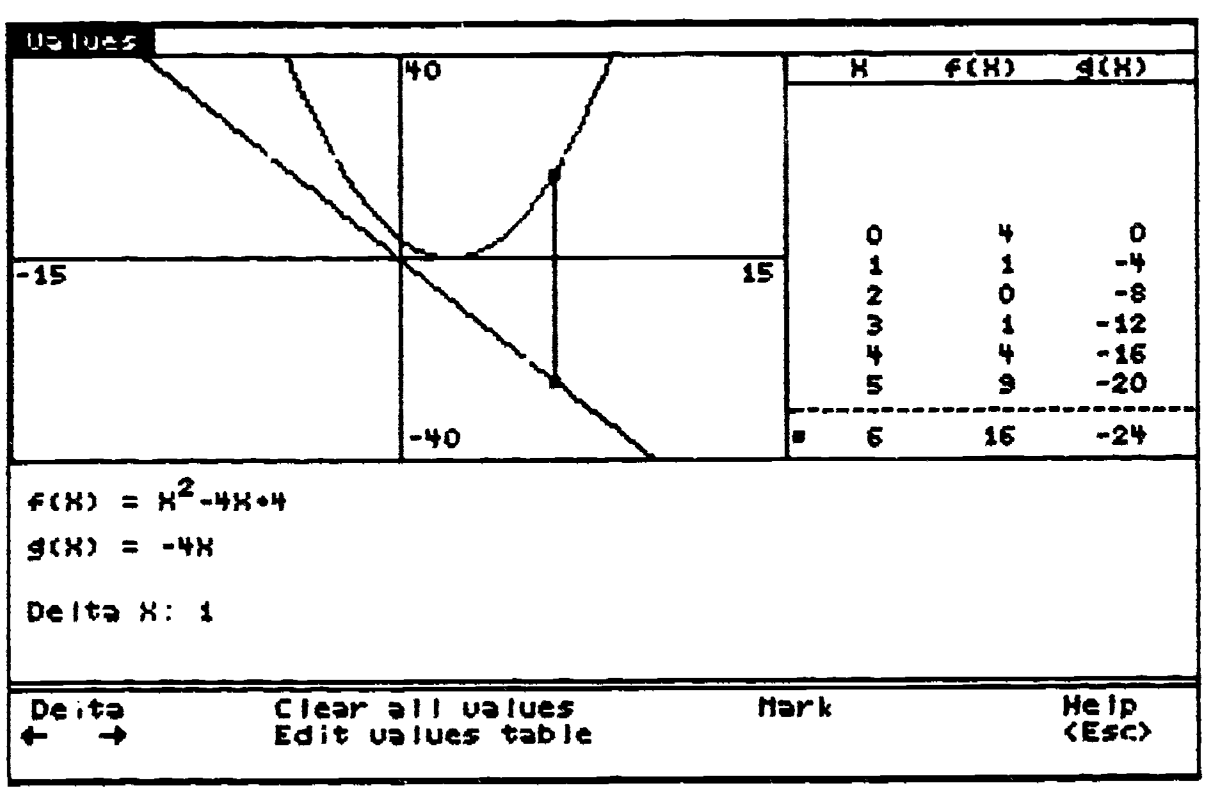


Figure 4. The display of function values.

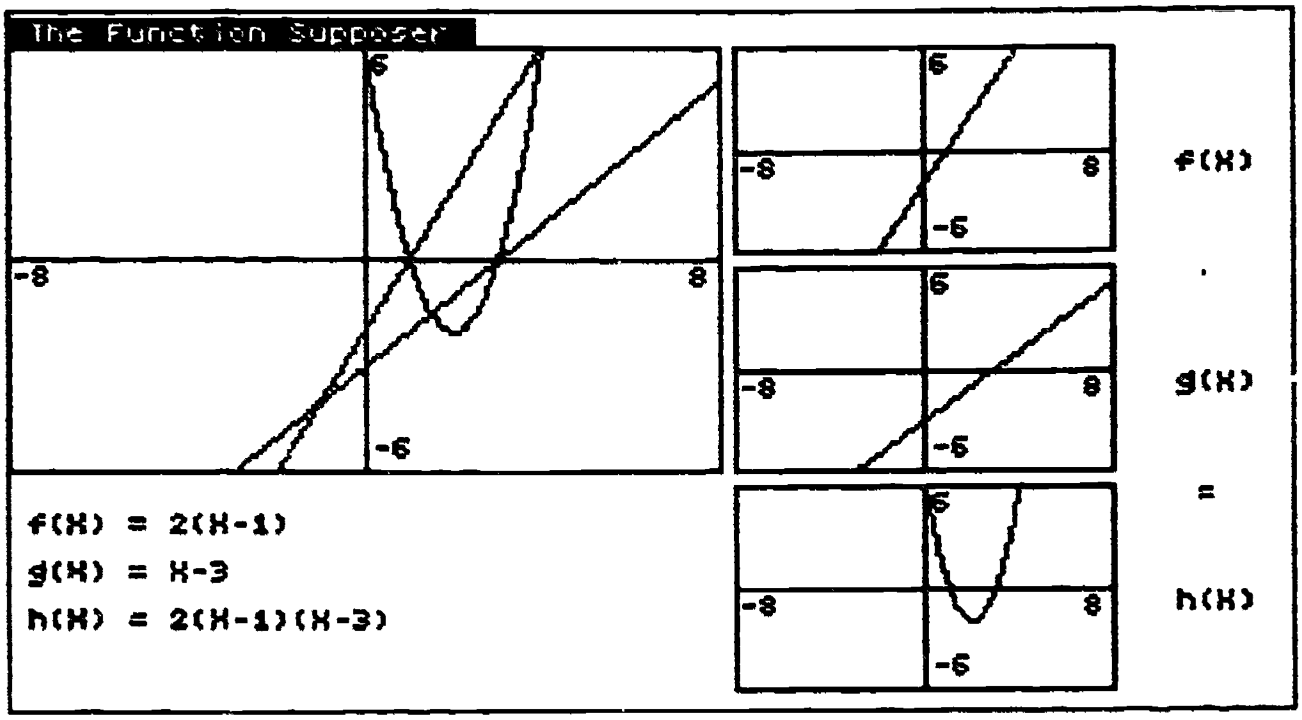
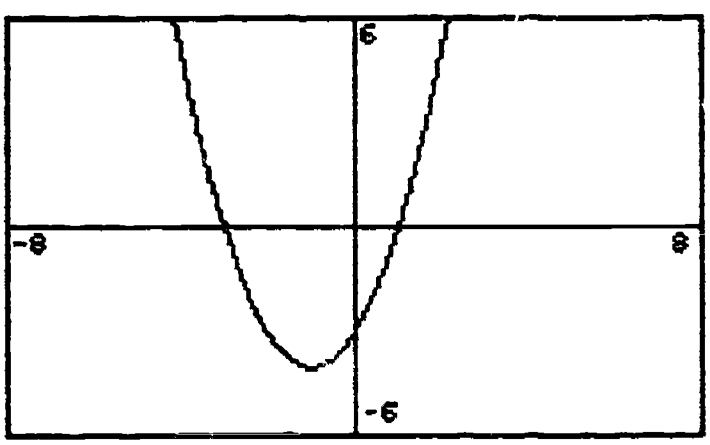


Figure 5. Building a new function in *The Function Supposer.*

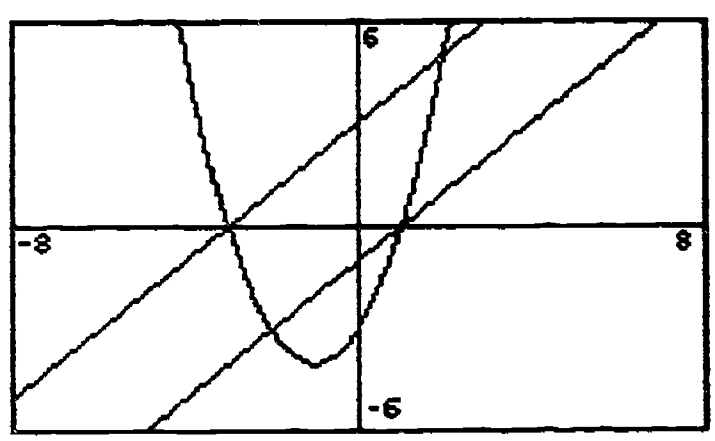
In the past, operations on functions were almost universally taught and learned in environments that made use of symbolic representations only. Solving equations, factoring polynomials, and finding roots of functions have traditionally been thought of as topics to be explored in a world of symbols. It is possible, however, to develop new understandings of these activities when they are examined in *The Function Supposer* environment.



f(x) = x^2+2x-3

Figure 6.

Consider the function in figure 6 (above). When students look only at its symbolic representation, $f(x) = x^2 + 2x - 3$, it is not immediately obvious to them that $(x-1)$ and $(x+3)$ are factors of this function. However, if they look at the graph of the function and understand the significance of the x intercepts, the factors are more evident because the graph crosses the x axis at $x = 1$ and $x = -3$. Even more convincing evidence can be derived by overlaying the graphs of the presumed factors (figure 7, below). Here is visual meaning for the fact that $(x-1)$ and $(x+3)$ are factors of this function. Note that the roots of $f(x)$ and $g(x)$ match those of $h(x)$. A function h represented as a product of functions f and g cannot have the value zero unless one of f or g has the value zero.



f(x) = x-1
g(x) = x+3
h(x) = (x-1)(x+3)

Figure 7.

Similarly, one can take a graphical approach to factoring polynomials and gain new insights into the behavior of functions. Consider, for

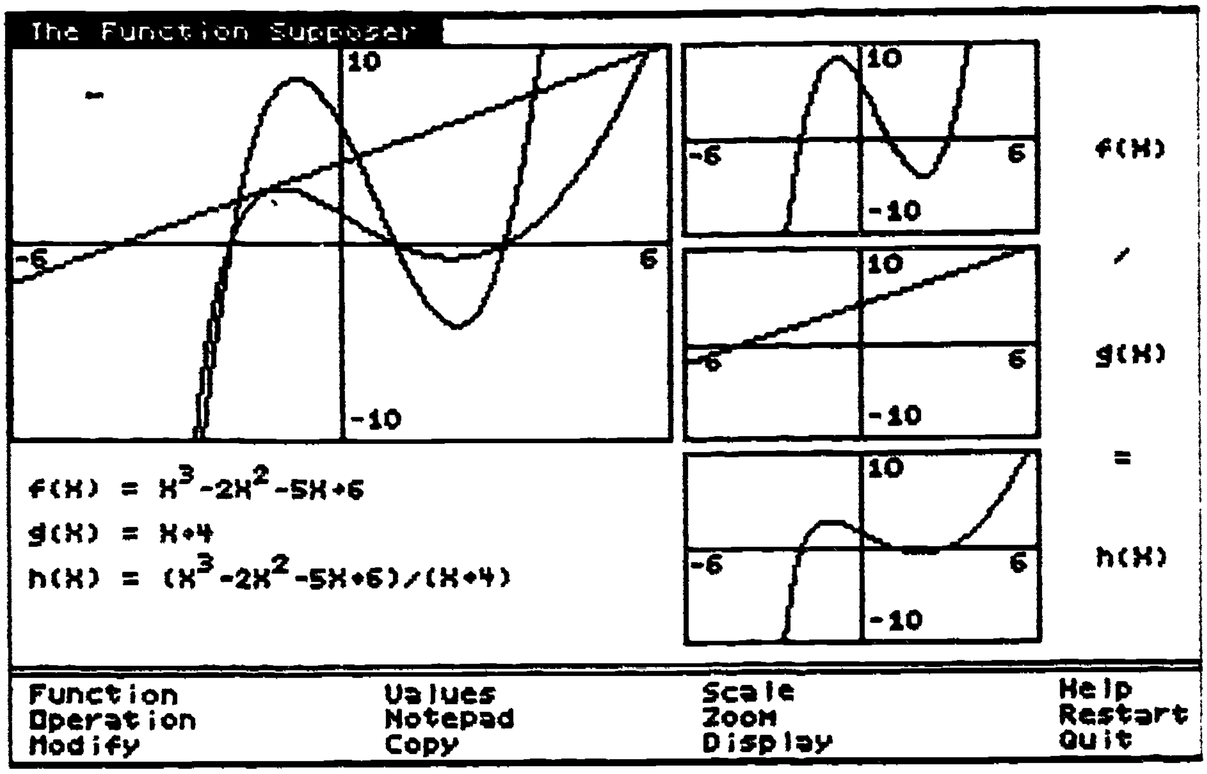


Figure 8. Dividing polynomials.

example, the function $f(x) = x^3 - 2x^2 - 5x + 6$. Figure 8 (above) shows what happens when one tries to "factor" this by $g(x) = x + 4$. It is interesting to note that the graph of the resulting function, *h*, has the same three roots as the original function, *f*. It is also noteworthy that the function *g* does not have a root in common with *f*. These and other features of these graphs may be obvious to those with more mathematical expertise, but such observations can be quite revealing for many students.

Another interesting exploration is to note the relationships among the shapes of the graphs, as well as their relative maxima and minima. Students come to expect that when a cubic polynomial is divided by a linear, the result ought to look something like a quadratic. But in the view shown in figure 8, the graph of $h(x)$ does not in fact look much like a quadratic. Students can be encouraged to explore the causes for the surprise appearance of $h(x)$; such exploration may lead them to conjecture that the scale in figure 8 is hiding some of the important detail of the graph of $h(x)$.

It is interesting to ask how these graphs might change as the divisor, $g(x)$, gets closer and closer to a factor of $f(x)$. Figure 9 (page 10) shows an image in which $g(x)$ is a true factor.

Notice again the relationships among the roots of the functions under consideration. And notice how the appearance of $h(x)$ changed from that in figure 8 (partially approximating a parabola as one looks further away from the root of $g(x)$ at $x = -4$)[4] to its parabolic appearance in figure 9.

The Function Comparator. Also important in the analysis of functions is an understanding of functional comparison. Existing curricula usually frame the study of functional comparison as the study of equations and inequalities, and use problems in this area (usually of the "solve for *x*" sort) to lead students to practice symbol manipulation. An example might be: "Find the set of all *x* that satisfy the comparison $x^2 - 2 > x$." Students familiar with such problems will successfully manipulate the symbols and find the solution set: $\{x | x < -1 \text{ or } x > 2\}$. While practice may increase students' computational fluency, it is not clear what insight they derive from learning to solve such problems. My colleague Judah Schwartz has proposed to keep the function concept central to such problems by reframing the problem as a comparison of two functions. In this example we would interpret the problem as the following comparison: $f(x) > g(x)$ where $f(x) = x^2 - 2$ and $g(x) = x$. Now,

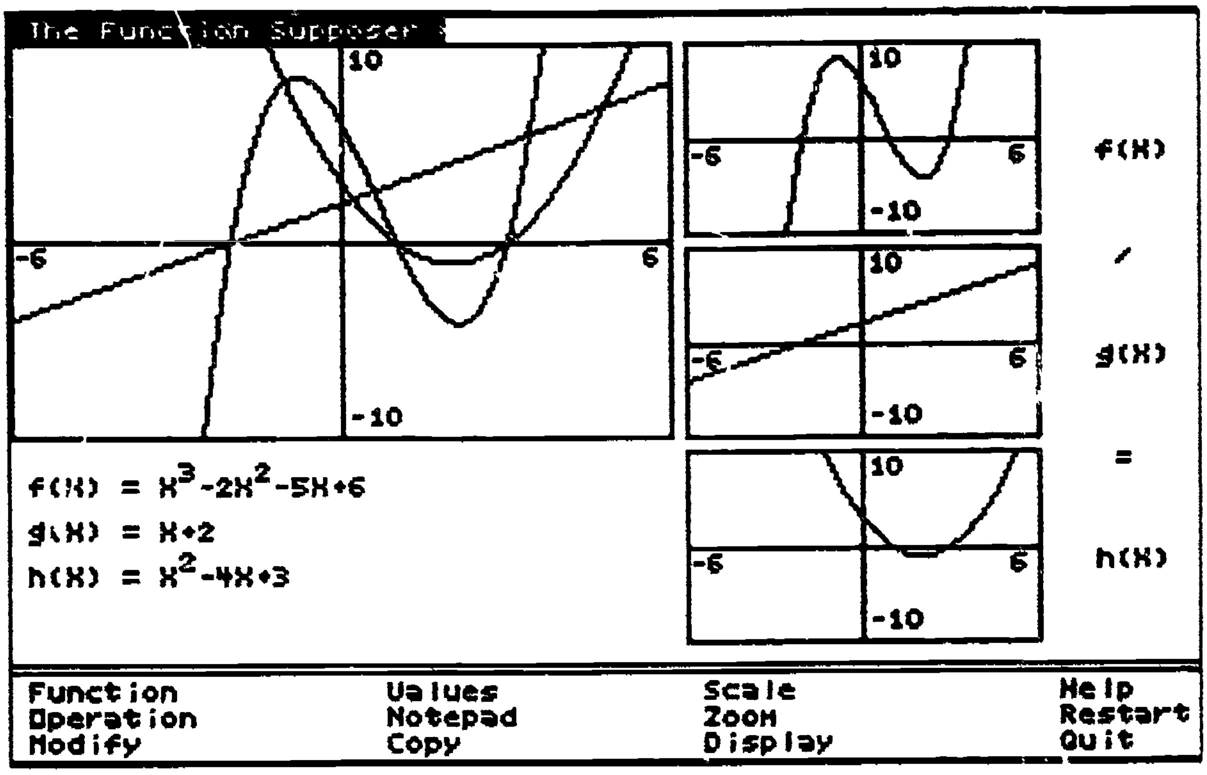


Figure 9. Factoring a cubic.

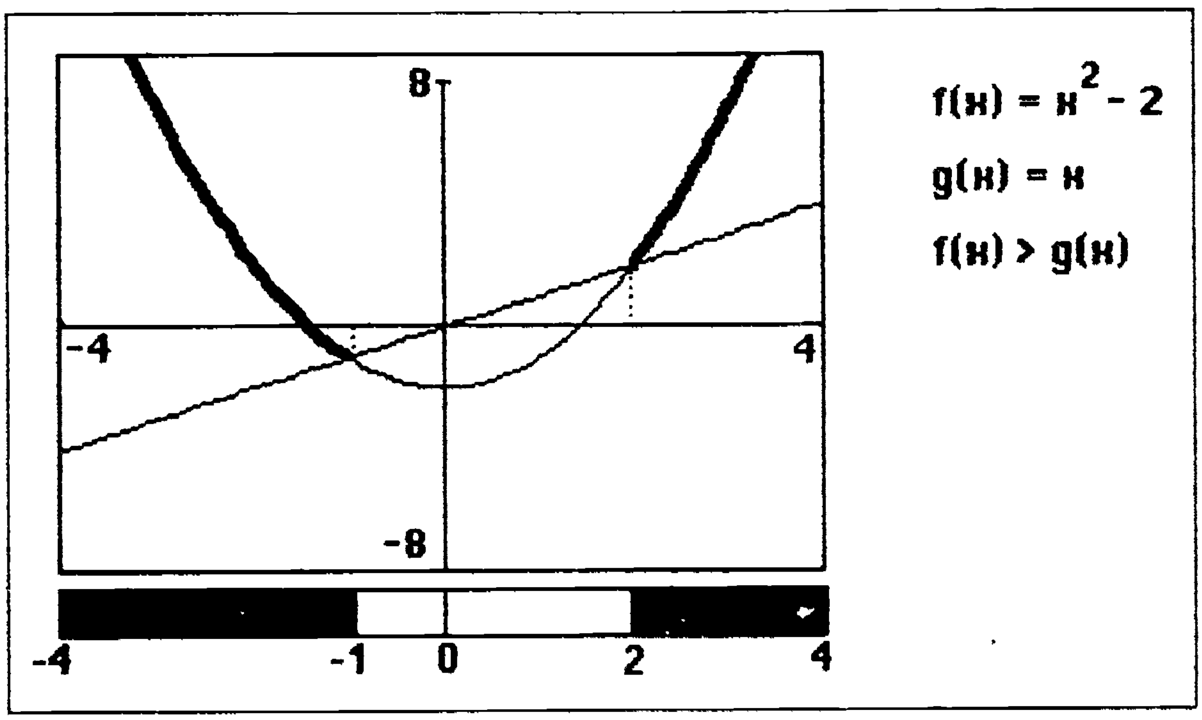


Figure 10. Comparing functions.

when we graph *f* and *g*, the solution set is visually represented and may become more understandable: it is the set of all *x* where the values of *f(x)* are greater than the values of *g(x)* (see figure 10, above).

Curriculum Materials for Exploring the Function Concept

Alongside the previously described software tools, we have developed a set of classroom materials that take advantage of the new opportunities for learning and teaching made available by the tools. We have developed these materials with the aid of teachers interested in encouraging their students to *build* mathematics ideas through group and individual project work. Our goal has been not only to engage students in interesting problem-solving activities, but also to engage students in interesting problem-*finding* activities. Mathematics is, after ail, more about learning to recognize worthwhile questions than it is about learning to solve questions that have already been answered. In designing these materials, we therefore asked, How can we encourage students to ask interesting mathematical questions and to defend and discuss their mathematical ideas?

Our approach has been to provide teachers with a variety of materials to motivate students to engage in and discuss mathematics in the classroom. The intent is to provide enough structure in the activity so that explorations carried out by students can be productive—that is, contain important mathematical content and also build on previously learned concepts and skills. At the same time, we want students to be raising at least as many questions based upon their own observations as are raised explicitly in the materials themselves. The following two examples suggest the range of materials we have been creating.[5]

Example 1: Exploring Transformation Points. The first project asks students to use their knowledge about functions and transformations to complete a table relating function expressions, critical points, transformations, and drawings of transformed function graphs. (See Table 1, page 12, for examples from such a table.) The objectives of this project are as follows:

- Explore how critical points in function graphs change through various graph transformations. For example, do the roots of functions change when the graph is stretched? translated?
- Investigate the similarities and differences in various graphical transformations.
- Explore how different classes of functions are affected by graphical transformations.
- Practice translating between the graphical and symbolic representations of functions.

The project first challenges students to fill in a table, using *The Function Analyzer* as a tool to assist them in their explorations, and then asks them to discuss their findings. The table provides sufficient structure for students to understand the challenge they face; the software provides sufficient support to help students find possible solutions for the task at hand; and the class discussion provides the opportunity for students to recognize that their particular line of thought was not necessarily the same as that of other students.

Commentary. This project can motiv stu-dents to clarify their knowledge of fu ons and transformations and requires them to think about transformations in different ways. It reinforces the relationships among the various representations of functions: as expressions, as tables of values, and as graphs.

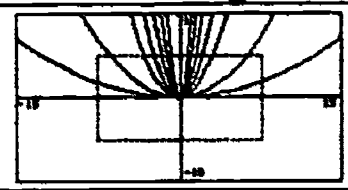
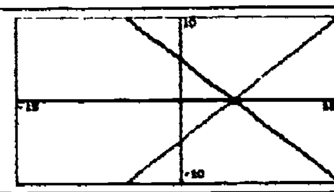
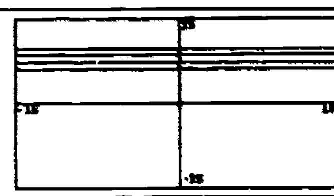
Notice also that the project is designed to allow many different possible solutions. In some cases, there are no possible answers (for example, in #2, there is no special invariant point when translating *-3x*, and similarly in #7). The project can lead a class into lively discussions in which students generalize the effects of transformations and the behavior of functions under various graphical transformations. In particular, students can explore how points on a graph are affected by translations, stretches, or flips and, most importantly, come to understand what aspects of a graph are invariant under such transformations.

Even more important, however, are the new questions students begin to raise (and then explore) as they compare their approaches. For example, in #6 students are provided with three roots and are asked to make a sketch and find an expression for a function having those three roots. As students compare their sketches, they find that not all are alike, but they also find that the sketches tend to have some similarities. The students' function expressions will also be different in some ways and similar in other ways. New questions are raised about how many possible solutions there may be and whether in fact there is a way (using parameters in the expression) to capture the entire family of solutions.[6]

Example 2: What's Missing? This second problem asks students to complete a simple table of calculations; but rather than using numbers (or even algebraic expressions) as the operands, students are asked to think about combining graphical objects. The goals of this project include the following:

- Explore how operations on functions, when viewed graphically, affect the shapes of the graphs.

Table 1. Example 1: Exploring Transformation Points

#	Function Expression	Points		Transformation	Drawings
		Value	Position		
1	x^2	(0,0)			
2	-3x			Translate	
3	$(x-2)(x+3)$		roots		
4		(0,-7)	y-intercept		
5		(5,0)			
6		(-5,0) (0,0) (7,0)		x-axis reflection	
7	10		any		

In the Points column, identify points on the function which do not change under the transformation listed or implied by the drawing.

- Learn to recognize transformations of graphs as operations on the functions represented by those graphs.
- Investigate properties of graphical transformations by experimenting with their effects on graphs.
- Practice reading and manipulating graphs of simple polynomials.

Students are asked to use their knowledge of functions and binary operations to help them find the missing element(s) in various graphs of functional relationships.

In the examples in Table 2 on page 14, students are asked to make predictions about the missing information before they check their predictions with *The Function Supposer.*

Commentary. This project is an unusual recasting of the kinds of problems that are often found in the present algebra curriculum. Consider how the project would look if we took a more traditional approach:

Solve for h(x)
Complete the calculations in the table below.

f(x)		g(x)		h(x)
$-x$	•	$x-4$	=	
x^2	+	2	=	
$x/2$	•	2	=	
x^3	–	x^3-2x	=	

In this form the intent of the problems is to have the student exercise symbolic manipulation skills—in effect, performing as an algebraic calculator. Whatever increased skill may come of such practice, little else is going on.

By contrast, in the visual alternative presented earlier, the variety of strategies for thinking about the problems—including strategies that make use of symbolic manipulations—keeps the problems rich. For example, a student might think about problem #1 ($f(x) \cdot g(x) = h(x)$) in the following way:

> The function h has to be zero when either f or g is zero, since h is the product of f and g. Therefore, h must be zero twice. Between these two places both f and g are negative; thus h must be positive. Given that h is the product of two linear functions, I will conjecture that the graph of h is an upside down parabola.

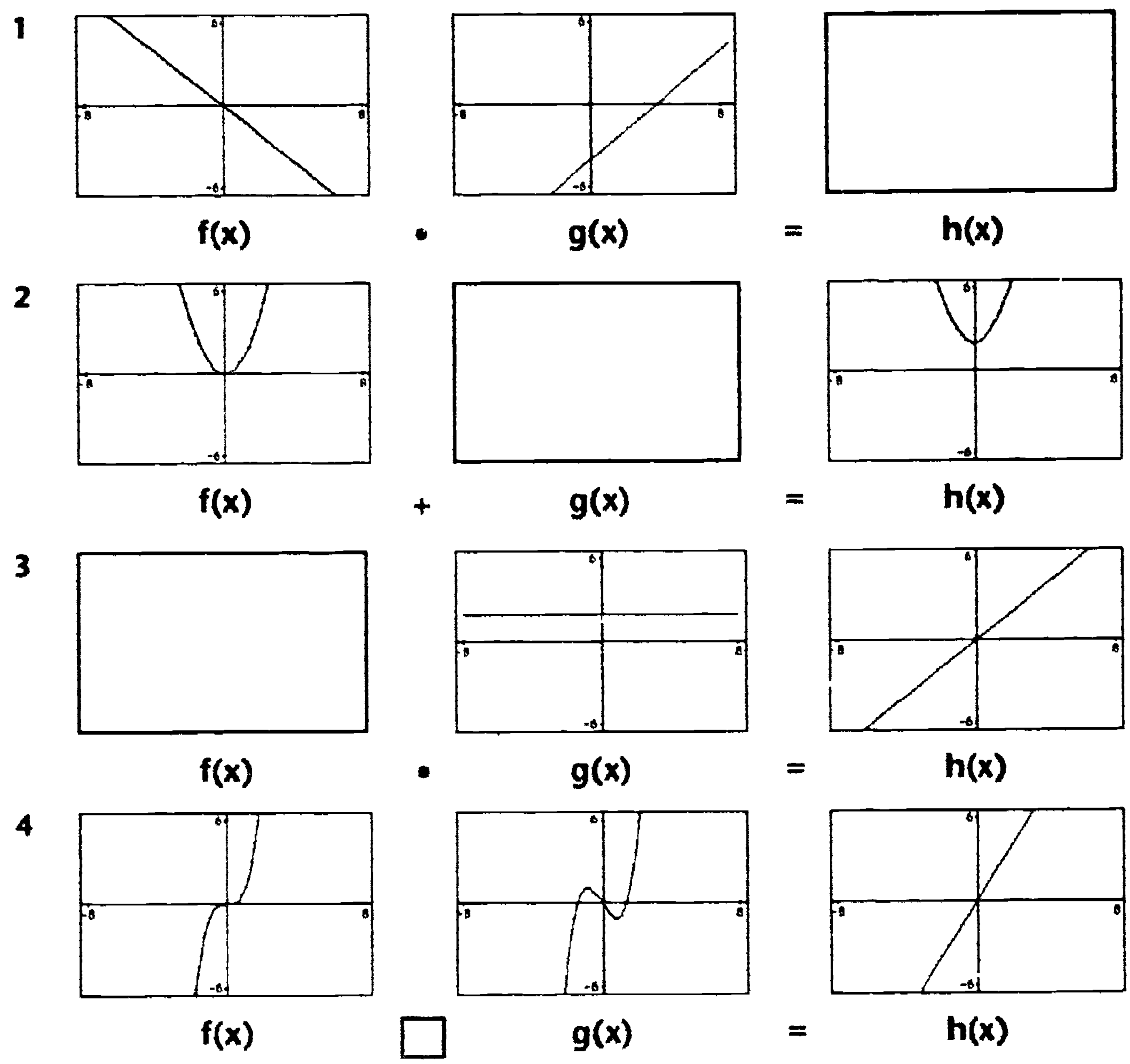
Problem #4 leads to even more interesting discussions and new questions. Some students will see that $f(x)$ is a cubic and that $h(x)$ is linear, and therefore conclude that division is necessary. But this reasoning presents a quandary because dividing a cubic by something to get a linear function requires that the divisor be quadratic. And $g(x)$ is certainly not quadratic. Other students may reason that if $h(x)$ is linear and $g(x)$ is cubic, then, given division as the operation, $f(x)$ must be a fourth-degree polynomial. Could this be possible? That in itself is an interesting discussion. And for all of these students some problem-posing activities are suggested: how could the problem be changed so that it can be solved with division as the operation? In fact, with the aid of *The Function Supposer*, some students will find that subtraction does indeed work as a solution to the problem.

New Directions

Our efforts to encourage and support curriculum reform in algebra will continue to focus on examining students' understandings, and misunderstandings, of algebraic concepts and how we might best use computer environments to provide more visual approaches to mathematics learning. But it is increasingly clear that reform of the algebra curriculum must be considered in the broader context of reform of the entire secondary mathematics curriculum. We see a strong connection between the dominance of syntactic/linguistic, non-visual mathematics instruction and the decontextualization of mathematics learning—the fragmentation of ideas, the answering of questions nobody has asked, the learning of discrete and disconnected facts and procedures, and the failure to see how one mathematical idea relates to another. Techniques out of context rarely mean much to students, and are hard to apply to novel problems. The consequence: students who don't like mathematics and students who cannot do much with it.

Of course, mathematical visualization is not, by itself, a solution. Tools for visualization are often made so context specific that they contribute to the isolation of mathematical ideas.

Table 2. Example 2: What's Missing?



Furthermore, visualization is not always easy. We argue, however, that much of the difficulty can be eliminated by building tools that provide students with the right kind of dynamic control over the visualizations they are using.

To this end, we are exploring an entirely new kind of software environment in which we introduce *controlled dynamic phenomena*. For example, we have been pursuing ways to develop students' understanding of function as mapping, and in particular, mappings from *R* to *R*. Most graphing software allows one to type in a function expression (for example, $f(x) = x + 2$) and then graphs the specified function in the real plane—that is, in *R*2. However, the dynamic-interactive properties of computer interfaces permit alternative graphical representations that offer different views of functional relationships. For example, instead of using the traditional perpendicular system of axes for graphing $f(x) = x + 2$, one can represent this mapping from *R* to *R* on two *parallel* number lines. With a single value of *x* plotted on one number line, and its image, $f(x)$, plotted on the other number line, a student can use a mouse to move the x-value on its axis, causing the image, $f(x)$, to move simultaneously on its parallel axis according to the functional relationship specified (see figure 11).[7]

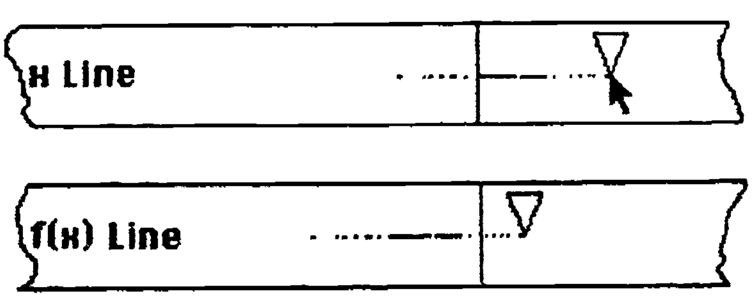


Figure 11. Mapping $x \cdot f(x) = x - 2$ using a parallel number line Dynagraph.

This *kinesthetic* approach to investigating functions is a catalyst for new intuitions and understandings. The simple function $f(x) = x - 2$ can be seen more obviously as a "subtracting of 2" from the value of x when investigated on the parallel axes than when graphed in the usual *R*2 plane. Various calculus concepts such as limits and rate of change become more directly observable. In fact, the behavior of functions with asymptotes, for example $f(x) = 1/(1-x)$, can support dramatically new intuitions about infinity. As a student moves the plotted x-value across the asymptote boundary, its image, $f(x)$, shoots off the screen towards positive infinity and then instantly back onto the screen from negative infinity. By moving the x-value back and forth across this boundary, one gets the distinct impression that positive and negative infinity are "connected back there somewhere," thereby encountering a rich topological idea.

The development of this kind of software allows us to ask some other kinds of questions: What do computers inspire us to do that we might not have done before? What new questions can be asked of students? What new options are available for the sequence in which algebraic skills or concepts are presented? What new content becomes interesting to teach? How do students misinterpret representations? Does student interaction with the representation affect these misinterpretations? What are the effects of allowing students to manipulate the graphic representations directly?

It is imperative that we focus our attention on how to engage more students in more mathematical inquiry and mathematical exploration in the classroom. Although the teacher is central to any such changes, we also need to provide teachers and students with new kinds of materials, new tools, and new approaches.

Teachers and students need projects and problems that lend themselves to a multitude of approaches and a variety of learning and teaching styles. In addition, students must feel that they can make progress on the project, can perceive avenues to explore the problems, and can find ways of assessing their progress. And once a solution to a problem is found, that should not necessarily be the end of the student's investigation—projects should lead not only to many different kinds of solutions, but also to other new projects.

That is, we want problems that lead more to investigations than to answers. All too often students work on problems that are best called exercises—activities designed to have students apply previously learned techniques to questions that yield single correct answers. When students have previously learned the relevant mathematics, they usually find such exercises simple; otherwise, they may find the exercises impossible. Although there is certainly a place for such mathematics exercises in the curriculum, this approach alone will instill in students only a dictionary knowledge of definitions, rules, and arbitrary tricks.

There is a need for much more research with individual students to help guide development efforts, but we are convinced that the story line for the algebra curriculum in grades 7 through 12 will ultimately have to be entirely rewritten. The current curriculum was conceived of during a time when paper and pencil were the only technologies available in the mathematics classroom. Now we are finding that with the use of computers and calculators we can set more ambitious goals than simply doing a better job at teaching the old curriculum. We can begin to formulate a mathematically rich and coherent approach to teaching and learning algebra.

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Notes

1. This series of four programs provides tools for introducing basic geometry concepts and for encouraging student investigation of the relationships among geometric shapes, elements, and constructions. *The Geometric Supposers* are designed for grades 8-12 and include the *preSupposer, Triangles, Quadrilaterals,* and *Circles*. A *Problems and Projects* curriculum guide is published for each program. Other available materials include teaching guides for integrating the software and project materials into commonly used textbook curricula and a series of videotapes by teachers. All materials are published by Sunburst Communications, Pleasantville, N.Y.

2. The research reported here was conducted under a subcontract from the Educational Technology Center, Harvard Graduate School of Education, and was funded in part by the United States Office of Educational Research and Improvement (Contract No. OERI 400-83-0041).

3. This message is detailed in Goldenberg, 1991.

4. This approximation to a parabola is even more apparent when the obscured portions of *h* are made visible at appropriate scales.

5. These materials are extracted from a number of publications now distributed by Sunburst Communications. These include *Visualizing Algebra: The Function Analyzer, The Function Supposer: Explorations in Algebra,* and *Problems and Projects for The Function Analyzer*. Credit for these materials goes to June Mark, Jim Hammerman, and Michal Yerushalmy.

6. In fact, all functions of the form $f(x) = a(x+5)x(x-7)$ have the three roots indicated. But while this family of functions captures all the cubic polynomials that have the three roots, there are infinitely many more functions—higher-order polynomials or non-polynomial functions—that also have those roots.

7. Credit for this idea of parallel number lines as an alternative to perpendicular axes goes to Phil Lewis.

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